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Integral Means of the Poisson Integral of a Discrete Measure

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It is proved that a function u, harmonic in the unit disc, can be represented in the form

$$u(z) = \sum \lambda_j \frac{1-|z|^2}{|1-w_j z|^2}, \qquad |z|<1,$$

with $|w_j| = 1$ $(j = 1, 2, ...)$, $\sum |\lambda_j|^p < \infty$, and $1/2 < p < 1$, if and only if

$$M_p(u, r) = O(1-r)^{1/p-1} \qquad (r \to 1^-).$$

The discussion of the case $p \leqslant 1/2$ involves the integral means of derivatives of u.

INTRODUCTION AND MAIN RESULT

The Poisson integral of a complex Borel measure λ on the unit circle T is the function u defined in the unit disc by

$$u(z) = \frac{1}{2\pi} \int_T \frac{1-|z|^2}{|1-\bar{w}z|^2}\, d\lambda(w).$$

By Herglotz' theorem [13, Theorem 11.19], the Poisson integral establishes an isomorphism between $M(T)$, the algebra of all complex Borel measures on T, and h^1, the algebra (relative to the convolution) of those complex-valued functions u harmonic in the disc, for which $\sup M_1(u, r) < \infty$ $(0 < r < 1)$. (We use M_p to denote the integral means of order p.) The fact that h^1 is an algebra is generalized in [11], where is it shown that, for $1/2 < p < 1$, the growth condition

$$M_p(u, r) = O(1-r)^{1/p-1} \qquad (r \to 1^-) \tag{1}$$

defines an infinite-dimensional subalgebra of h^1; denote it by A_0^p. Hence two problems arise naturally: extend the scale $\{A_0^p\colon 1/2 < p < 1\}$ to the

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values $p \leqslant 1/2$, and represent A_0^p as a subalgebra of $M(T)$. Our main result is a solution to these problems.

THEOREM. *Let* $0 < p < 1$ *and* n *be an integer* $> 1/p - 1$. *For a function* u, *harmonic in the unit disc, the following conditions* (A) *and* (B) *are equivalent.*

(A) (i) $M_p(D^{n-1}u, r) = O(1-r)^{1/p-n}$ $(r \to 1^-)$, *and*

(ii) $\lim_{r \to 1} u(re^{i\theta}) = 0$ *for almost all* θ,

(B) u *is the Poisson integral of a discrete measure* λ *such that*

$$\sum_{w \in T} |\lambda\{w\}|^p < \infty. \tag{2}$$

If in addition $p > 1/2$, then (B) is equivalent to (1) as well.

Here $D^{n-1}u = \partial^{n-1}u/\partial\theta^{n-1}$ for $n \geqslant 2$, and $D^0u = u$.

A formal consequence of the theorem is that the class A_0^p, $0 < p < 1$, of functions satisfying (A) is independent of n for $n > 1/p - 1$. The Poisson integral establishes an isomorphism between A_0^p and $M_p(T)$, the class of discrete measures satisfying (2). That $M_p(T)$ is an algebra is easily deduced from the fact that the convolution of two discrete measures is discrete and

$$(\lambda * \mu)\{w\} = \sum_{a \in T} \lambda\{a\}\, \mu\{\bar{a}w\}, \qquad w \in T.$$

That condition (B) implies (A) follows from Shapiro's estimates for derivatives of the Poisson kernel; see Lemma 1.7. In order to prove that (A) implies (B) we use the notion of bounded p-variation introduced by Musielak and Orlicz [7]. For $0 < p < 1$ condition (B) is equivalent to the following (see Lemmas 2.2 and 2.3):

(C) u is the Poisson–Stieltjes integral of a function of bounded p-variation.

In Section 1 we define some classes of harmonic functions and list some well known results of Hardy and Littlewood. The main result is that, for $1/2 < p < 1$ and $n > 1/p - 1$, condition (A) is equivalent to (1) (Lemma 1.8).

In Section 2 we extend the Herglotz theorem by proving that, for $1/2 < p \leqslant 1$, condition (1) is equivalent to (C) (Theorem 2.6), which proves the case $p > 1/2$ of our main result.

In Section 3 we finish the proof. First we use the results of Sections 1 and 2 to conclude that a function satisfying (A) is the Poisson integral of a discrete measure. To prove then that this measure satisfies (B) we use some facts from the theory of Lipschitz spaces.

Remarks suggested by the referee. In the next section we use the condition obtained from (A) by replacing (ii) with $\liminf |u(re^{i\theta})| = 0$ (a.e), and we prove that it implies (B). On the other hand, by the Main Theorem, condition (B) implies (A) in a stronger form, namely, $\lim u(re^{i\theta})=0$ for $[0, 2\pi]\backslash S$, where S is a countable set.

It follows from the "increasing property" of the integral means of holomorhic functions that a nontrivial holomorphic function cannot satisfy condition (1) (for $p<1$). We conjecture that if $p \leqslant 1/2$, the same holds in the class of harmonic functions. Namely, our proof of Theorem 2.6 shows that (1) implies (B) for $p \leqslant 1/2$ as well, and we hope that if u satisfies (B), then $M_p(u, r)$ behaves like $M_p(P, r)$, where P is the Poisson kernel. A routine calculation shows that the Poisson kernel does not satisfy (1) for $p \leqslant 1/2$. (See [12], for example.)

1. Lemmas on Harmonic Functions

Let $h(\Delta)$ denote the class of all complex-valued functions harmonic in the open disc Δ of the complex plane. Each $u \in h(\Delta)$ has a unique series expansion

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} \hat{u}(j)\, r^{|j|} e^{ij\theta}$$

which converges uniformly and absolutely on compact subsets of Δ. In particular, if P is the Poisson kernel,

$$P(z) = \operatorname{Re} \frac{1+z}{1-z} = \frac{1-r^2}{1+r^2-2r\cos\theta} \qquad (z = re^{i\theta}),$$

then $\hat{P}(j)=1$ for all j.

The convolution, $u * v$, of two elements of $h(\Delta)$ is defined by

$$(u * v)^{\wedge}\,(j) = \hat{u}(j)\,\hat{v}(j) \qquad (j = 0, \pm 1, ...).$$

The class $h(\Delta)$, treated as a linear space in the standard way, is an algebra relative to the operation $*$, with Poisson's kernel as the unit. In this section we define some subalgebras of $h(\Delta)$.

1.1. *The Class* $h^p(\alpha)$

Let $0<p\leqslant 1$. For a real number α let

$$h^p(\alpha) = \{u \in h(\Delta) : M_p(u, r) = O(1-r)^{\alpha}\},$$

where as usual

$$M_p(u, r) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^p\, d\theta\right\}^{1/p}.$$

The following theorem is only a reformulation of the classical results of Hardy and Littlewood [4, 5]. Here

$$(D^n u)(re^{i\theta}) = \partial^n u/\partial\theta^n \qquad (n = 0, 1, \ldots).$$

1.2. THEOREM. *For a real number α the following hold.*

(a) $h^p(\alpha) \subset h^q(\alpha + 1/q - 1/p)$ *for* $p < q$.

(b) *If* $u \in h^p(\alpha)$, *then* $D^1 u \in h^p(\alpha - 1)$.

(c) *For* $\alpha < 0$, $u \in h^p(\alpha)$ *iff* $D^1 u \in h^p(\alpha - 1)$.

(d) *For* $\alpha < 0$, *if u is in* $h^p(\alpha)$, *then so is its harmonic conjugate.*

For further information and references we refer the reader to [2]. The proof of this theorem as well as of other similar results can be based on the following lemma of Hardy and Littlewood [4].

1.3. LEMMA. *Let* $0 < p < \infty$ *and* $0 < \varepsilon < 1$. *If* $u \in h(\Delta)$, *then*

$$|u(z)|^p \leqslant C(1 - |z|)^{-2} \int_{\Delta_z} |u|^p \, dm,$$

where $\Delta_z = \{w : |w - z| < \varepsilon(1 - |z|)\}$ *and dm denotes the Lebesgue measure; C is a constant depending only on p and ε.*

A short proof in a more general context can be found in [10].

1.4. *The Algebra* A^p

For $0 < p \leqslant 1$ we define the class

$$A^p = \{u \in h(\Delta) : D^n u \in h^p(1/p - 1 - n)\},$$

where n is an arbitrary integer $> 1/p - 1$. Theorem 1.2(c) guarantees that this definition is independent of n.

That A^p is actually an algebra can be deduced from [11, Theorem 3.2]: If $u, v \in h^p(\alpha)$, then $u * v \in h^p(2\alpha + 1 - 1/p)$.

Indeed, if $D^n u$ and $D^n v$ are in $h^p(1/p - 1 - n)$, $n > 1/p - 1$, then

$$D^{2n}(u * v) = (D^n u) * (D^n v) \in h^p(1/p - 1 - 2n)$$

and hence $u * v \in A^p$.

1.5. PROPOSITION. *If* $p < q \leqslant 1$, *then* $A^p \subset A^q$.

Proof. Let $n > 1/p > 1/q$. If $D^{n-1} u \in h^p(1/p - n)$, then $D^{n-1} u \in h^q(1/q - n)$, by Theorem 1.2(a). ∎

1.6. *The algebra* A_0^p

Let $A_0^p = h^p(1/p - 1)$ for $1/2 < p < 1$. If $p \leqslant 1/2$, let n be the unique integer such that $1/p - 1 < n \leqslant 1/p$ and define

$$A_0^p = \{u\colon D^{n-1}u \in h^p(1/p - n) \text{ and } u \in h_0(\Delta)\},$$

where

$$h_0(\Delta) = \{u \in h(\Delta) : \liminf_{r \to 1_-} |u(re^{i\theta})| = 0 \text{ a.e.}\}.$$

Observe that $h^p(1/p - 1) \subset h_0(\Delta)$, by Fatou's lemma ($p < 1$).

1.7. LEMMA. *The Poisson kernel is in* A_0^p *and* $A_0^p \subset A^p$.

Proof. That $P \in h_0(\Delta)$ is verified immediately. In [14] Shapiro proved that

$$|(D^{n-1}P)(z)| \leqslant C_n \frac{1 - |z|}{|1 - z|^{n+1}},$$

hence

$$M_p^p(D^{n-1}P, r) \leqslant C_{n,p}(1 - r)^p \int_0^{2\pi} |1 - re^{i\theta}|^{-(n+1)p}\, d\theta.$$

Using the familiar estimate for the integral [1, p. 65], we conclude that, for $n > 1/p - 1$,

$$M_p^p(D^{n-1}P, r) \leqslant C_{n,p}(1 - r)^{p - (n+1)p + 1},$$

and hence $D^{n-1}P \in h^p(1/p - n)$.

The inclusion $A_0^p \subset A^p$ is a consequence of Theorem 1.2(b). ∎

1.8. LEMMA. $A^p \cap h_0(\Delta) \subset A_0^p$ *for* $1/2 < p < 1$.

Proof. Assume, without loss of generality, that u is the real part of a holomorphic function f. If u is in A^p, then so is f, by Theorem 1.2(d) and the definition of A^p; i.e.,

$$M_p^p(D^1 f, r) = O(1 - r)^{1 - 2p}. \qquad (+)$$

For $f \in A^p$ let $g_k(\theta) = f(r_k e^{i\theta})$, where $r_k = 1 - 2^{-k}$ $(k = 1, 2, ...)$, and $f_r(\theta) = f(re^{i\theta})$, $0 < r < 1$. By an application of the Hardy–Littlewood complex maximal theorem we find that

$$\|f_\rho - f_r\|_p \leqslant C_p(\rho - r)\, M_p(D^1 f, \rho) \qquad (r < \rho < 1) \tag{1}$$

(see [6], for example), where $\| \;\|_p$ stands for the "norm" in $L^p(0, 2\pi)$. In particular, by $(+)$,

$$\|g_k - g_{k-1}\|_p^p \leqslant C_p 2^{-k(1-p)}.$$

This implies that f belongs to the Hardy space H^p (because $1 - p > 0$; see also Theorem 3.3 below) and

$$\begin{aligned}\|f^+ - g_k\|_p^p &\leqslant \sum_{j=k+1}^{\infty} \|g_j - g_{j-1}\|_p^p \\ &\leqslant C_p 2^{-k(1-p)},\end{aligned}$$

where

$$f^+(\theta) = \lim_{r \to 1-} f(re^{i\theta}) \qquad \text{a.e.}$$

(For information on the boundary values see [1].) In other words,

$$\|f^+ - f_r\|_p \leqslant C_p(1 - r)^{1/p - 1} \tag{2}$$

if $r = r_k$ for some integer k. If $r < 1$ is arbitrary, we choose k so that $r_{k-1} \leqslant r \leqslant r_k$ and use (1) with $\rho = r_k$ to conclude that (2) holds for all $r < 1$.

Finally, if $u = \operatorname{Re} f$ is in $h_0(\Delta)$, then $\operatorname{Re} f^+ = 0$ and, by (2),

$$M_p(u, r) = \|u_r\|_p = O(1 - r)^{1/p - 1},$$

i.e., $u \in h^p(1/p - 1) = A_0^p$, which was to be proved. ∎

1.9. Lemma. *If* $q \leqslant 1/2 < p < 1$, *then* $A^q \cap h_0(\Delta) \subset A_0^p \subset h^1 := h^1(0)$.

Proof. By Proposition 1.5 and Lemma 1.8,

$$A^q \cap h_0(\Delta) \subset A^p \cap h_0(\Delta) \subset A_0^p = h^p(1/p - 1)$$

for $q \leqslant 1/2 < p < 1$. By Theorem 1.2(a), $h^p(1/p - 1) \subset h^1(0)$. ∎

2. Functions of Bounded p-Variation

A partition Π of the interval $[0, 2\pi]$ is a finite subset of $[0, 2\pi]$ containing the end points. For a complex valued function ϕ on $[0, 2\pi]$ we define the quantities

$$V_p(\phi, \Pi) = \sum_{j=0}^{n} |\phi(\theta_{j+1}) - \phi(\theta_j)|^p,$$

where $\Pi = \{\theta_j : 0 \leqslant j \leqslant n\}$ and $\theta_j < \theta_{j+1}$ for all j. Following Musielak and Orlicz [7] we say that ϕ is of bounded p-variation if

$$V_p(\phi) := \sup \{V_p(\phi, \Pi) : \Pi \text{ partition}\} < \infty.$$

If $p = 1$, this notion reduces to the notion of bounded variation in the usual sense.

2.1. *The Class NBV_p*

This class consists of all left-continuous functions ϕ of bounded p-variation on $[0, 2\pi]$ with $\phi(0) = 0$. It is easily verified that $NBV_p \subset NBV := NBV_1$ for $p < 1$. Therefore for $\phi \in NBV_p$ the step function is well defined in the following way:

$$s_\phi(0) = 0,$$

$$s_\phi(\theta) = \sum_{x=0}^{\theta} [\phi(x^+) - \phi(x)], \qquad 0 < \theta \leqslant 2\pi.$$

The sum is taken over the points of discontinuity of ϕ in $[0, \theta)$.

2.2. Lemma. *If $\phi \in NBV_p$, $0 < p < 1$, then $s_\phi \in NBV_p$ and*

$$V_p(\phi) \geqslant V_p(s_\phi) = \sum_{x=0}^{2\pi^-} |\phi(x^+) - \phi(x)|^p.$$

Proof. Let S be a finite subset of $[0, 2\pi)$. Choose $\delta > 0$ so that the intervals $[x, x + \delta)$, $x \in S$, are pairwise disjoint. Then

$$V_p(\phi) \geqslant \sum_{x \in S} |\phi(x + \varepsilon) - \phi(x)|^p$$

for all ε, $0 < \varepsilon < \delta$. This implies that

$$V_p(\phi) \geqslant \sum_{x \in S} |\phi(x^+) - \phi(x)|^p,$$

and hence, because S is arbitary,

$$V_p(\phi) \geqslant \sum_{x=0}^{2\pi^-} |\phi(x^+) - \phi(x)|^p.$$

To prove the rest let $\Pi = \{\theta_j\}$ be a partition of $[0, 2\pi]$. Then

$$\begin{aligned} V_p(s_\phi, \Pi) &= \sum_j \left| \sum_{x=\theta_j}^{\theta_{j+1}^-} [\phi(x^+) - \phi(x)] \right|^p \\ &\leqslant \sum_j \sum_{x=\theta_j}^{\theta_{j+1}^-} |\phi(x^+) - \phi(x)|^p \\ &= \sum_{x=0}^{2\pi^-} |\phi(x^+) - \phi(x)|^p. \end{aligned}$$

This shows that $s_\phi \in NBV_p$ and $V_p(s_\phi) \leqslant V_p(\phi)$ for $\phi \in NBV_p$. Finally, if we apply the first part of the proof to s_ϕ and use the fact that $s_\phi(x^+) - s_\phi(x) = \phi(x^+) - \phi(x)$, we see that $V_p(s_\phi) \geqslant \sum \cdots$. ∎

2.3. Lemma [7]. *If $\phi \in NBV_p$ for some $p < 1$, then $\phi = s_\phi$.*

Proof. Let ϕ be real valued. If $\phi \in NBV_p$, then $f := \phi - s_\phi$ is continuous and belongs to NBV_p, by Lemma 2.2. Assume that $f(x) \neq 0$ for some x. Then, for each fixed integer $n > 0$, we choose an increasing sequence $\{t_j\}_0^n$, $t_o = 0$, $t_n = x$, such that $f(t_{j+1}) - f(t_j) = f(x)/n$. Since

$$\sum_{j=0}^{n} |f(t_{j+1}) - f(t_j)|^p = n^{1-p} |f(x)|^p,$$

we find that $|f(x)|^p \leqslant n^{p-1} V_p(f) \to 0$ $(n \to \infty)$. This proves the lemma. ∎

2.4. *Partitions of Class $\mathscr{P}_n$*

For $n = 1, 2, ...$, let $\mathscr{P}_n$ denote the class of partitions $\Pi = \{\theta_j\}$ such

$$2^{-n-1} < (\theta_{j+1} - \theta_j)/2\pi < 2^{-n}$$

for all j. The following assertions are easily verified.

(a) $2^n < \operatorname{card} \Pi \leqslant 2^{n+1}$ for $\Pi \in \mathscr{P}_n$.

(b) If Π is an arbitrary partition, then there is a partition Π' of class $\mathscr{P}_n$, for some n, such that $\Pi' \supset \Pi$.

(c) If $\Pi' \in \mathscr{P}_n$, then there is a $\Pi'' \in \mathscr{P}_{n+1}$ such that $\Pi'' \supset \Pi'$.

2.5. Lemma. *A function $\phi \in NBV$ is of bounded p-variation if there are a sequence $\{\phi_n\}$, constant $C < \infty$ independent of n and a dense subset S of $[0, 2\pi]$ such that*

(i) $V_p(\phi_n, \Pi) \leqslant C$ *for every* $\Pi \in \mathscr{P}_n$, *and*

(ii) $\lim_n \phi_n(\theta) = \phi(\theta)$ *for* $\theta \in S$.

Proof. Assume first that $\Pi = \{\theta_j\}_0^k$ is a partition such that

(iii) $\Pi \setminus \{0, 2\pi\} \subset S$.

By 2.4, (b) and (c), we find an integer N and a sequence $\{\Pi_n\}$ of partitions such that $\Pi_n \in \mathscr{P}_n$ and $\Pi \subset \Pi_n$, $n > N$. Hence, using (i) and the easily verified fact that $V_p(\psi, \Pi') \leqslant V_P(\psi, \Pi'')$ whenever $\Pi' \subset \Pi''$,

$$V_p(\phi_n, \Pi) \leqslant V_p(\phi_n, \Pi_n) \leqslant C.$$

And hence, by (ii) and (iii),

$$\sum_{j=1}^{k-2} |\phi(\theta_{j+1}) - \phi(\theta_j)|^p \leqslant C,$$

which implies that $V_p(\phi, \Pi) \leqslant C_1$, where C_1 is independent of Π.

Let $\Pi' = \{\theta_j'\}_0^k$ be an arbitrary partition and $\varepsilon > 0$. Since ϕ is left-continuous and S is dense, there is a partition $\Pi = \{\theta_j\}_0^k$ satisfying (iii) and such that

$$|\phi(\theta_j) - \phi(\theta_j')|^p \leqslant \varepsilon/2k \qquad \text{for} \quad j = 0, ..., k.$$

This implies that $V_p(\phi, \Pi') \leqslant V_p(\phi, \Pi) + \varepsilon$, which concludes the proof. ∎

2.6. THEOREM. *Let* $1/2 < p \leqslant 1$. *A function u, harmonic in the unit disc, belongs to the class* $h^p(1/p - 1)$ *if and only if it is the Poisson–Stieltjes integral of a function of bounded p-variation.*

Remark. In the case where $p = 1$ this result is a reformulation of Herglotz' theorem because $h^1(0) = h^1$.

Proof. Let $\phi \in NBV_p$, $p < 1$, and

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t)\, d\phi(t), \tag{1}$$

the Poisson–Stieltjes integral of ϕ. By Lemmas 2.2 and 2.3,

$$u(z) = \sum_0^\infty a_k P(z\bar{w}_k) \qquad \text{for} \quad |z| < 1,$$

where $|w_k| = 1$ and $\sum |a_k|^p < \infty$. Hence

$$M_p^p(u, r) \leqslant \sum |a_k|^p M_p^p(P, r), \qquad 0 < r < 1,$$

which implies that $u \in h^p(1/p - 1) = A_0^p$ because of Lemma 1.7.

Let $u \in h^p(1/p-1)$, $1/2<p<1$. Then, by Lemma 1.9 and Herglotz' theorem, there is a function $\phi \in NBV$ such that (1) holds. By partial integration,

$$u(re^{i\theta}) = \phi(2\pi)\, P(r, \theta)/2\pi + \int_0^{2\pi} \phi(t)\, D^1 P(r, \theta - t)\, dt/2\pi,$$

which can be written as

$$u(re^{i\theta}) = \text{const. } P(r, \theta) + D^1 U(re^{i\theta}),$$

where

$$U(re^{i\theta}) = U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t)\, \phi(t)\, dt.$$

Hence

$$M_p^p(D^1 U, r) = O(1-r)^{1-p} \tag{2}$$

because u and P are in $h^p(1/p-1)$.

We use (2) and Lemma 2.5 to prove that $\phi \in NBV_p$. Let

$$\phi_n(\theta) = U(r_n, \theta), \qquad r_n = 1 - 2^{-n}.$$

It is well known (see [12], for example) that $\lim \phi_n(\theta) = \phi(\theta)$ a.e. Thus we have to prove the condition 2.5(i) is satisfied. Let $\Pi = \{\theta_j\}_0^N$ be a partition of class $\mathscr{P}_n$ (see 2.4), where n is a fixed integer. Then

$$|\phi_n(\theta_{j+1}) - \phi_n(\theta_j)| = (\theta_{j+1} - \theta_j)\, |D^1 U(r_n, \theta_j')|$$

for some $\theta_j' \in [\theta_j, \theta_{j+1}]$ $(0 \leqslant j \leqslant N-1)$. Hence, by Lemma 1.3,

$$|\phi_n(\theta_{j+1}) - \phi_n(\theta_j)|^p \leqslant C_p 2^{-np+2n} \int_{\Delta_{n,j}} |D^1 U|^p\, dm,$$

where

$$\Delta_{n,j} = \{re^{i\theta} : r_{n-1} < r < r_{n+1}, \theta_{j-1} < \theta < \theta_{j+2}\}$$

for $0 \leqslant j \leqslant N-1$, and $\theta_{-1} = -2\pi/2^n$, $\theta_{N+1} = 2\pi + 2\pi/2^n$. Hence, by summation from $j=0$ to $N-1$, we find that

$$V_p(\phi_n, \Pi) \leqslant C_p 2^{n(2-p)} \int_{\Delta_n} |D^1 U|^p\, dm,$$

where

$$\Delta_n = \{z : r_{n-1} < |z| < r_{n+1}\},$$

and C_p depends only on p. Finally, by integration in polar coordinates and using (2), we see that $V_p(\phi, \Pi) \leqslant C$, where C is independent of n. This completes the proof. ▮

3. Proof of the Main Result

In view of the results of Sections 1 and 2 our main result can be stated as follows.

3.1. Theorem. *Let $0 < p < 1$. For a function $u \in h(\Delta)$ the following conditions are equivalent.*

(i) $u \in A_0^p$.

(ii) $u \in A^p \cap h_0(\Delta)$.

(iii) *u is the Poisson–Stieltjes integral of a function of bounded p-variation.*

In the case $p > 1/2$ the theorem follows from Theorem 2.6 and the equality $A_0^p = A^p \cap h_0(\Delta)$ (Lemmas 1.7 and 1.8). In order to complete the proof we need a fact concerning Lipschitz spaces.

3.2. *Symmetric Differences*

For a complex valued function g, defined on the real line, let $\Delta_t^n g$ denote the nth symmetric difference with step t:

$$(\Delta_t^1 g)(\theta) = g(\theta + t) - g(\theta),$$
$$\Delta_t^n g = \Delta_t^1 \Delta_t^{n-1} g \qquad (n = 2, 3, ...).$$

For fixed n and t, Δ_t^n is a linear operator which preserves L^p spaces.

We need the following theorem of Oswald [8] and Gwiliam [3]. For a simple proof see [9].

3.3. Theorem. *Let $0 \leqslant p \leqslant 1$ and $0 < \alpha < n$, where n is an integer. For a holomorphic function f, $D^n f$ is in $h^p(\alpha - n)$ if and only if f is in H^p and the boundary function f^+ satisfies the condition*

$$\|\Delta_t^n f^+\|_p = O(t^\alpha) \qquad (t \to 0+).$$

COROLLARY. *Let* $0<p<1$ *and* n *be a positive integer* $>1/p-1$. *A holomorphic function* f *belongs to* A^p *if and only if* f *belongs to* H^p *and*

$$\|\Delta_t^n f^+\|_p = O(t^{1/p-1}) \qquad (t \to 0^+).$$

Let us note that this corollary does not hold for $p=1$.

LEMMA. *A left-continuous function* ϕ *of bounded q-variation* $(q<1)$ *on* $[0, 2\pi]$ *is of bounded p-variation if (and only if)*

$$\sum_{x=0}^{2\pi-} |\phi(x+)-\phi(x)|^p < \infty.$$

Proof. If $\phi \in NBV_q$ $(q<1)$, then $\phi = s_\phi$, by Lemma 2.3. Now the result follows from the proof of Lemma 2.2. ∎

3.5. *Proof of Theorem* 3.1. That (iii) implies (i) is deduced from Lemma 1.7 as in the proof of Theorem 2.6; we only have to observe that the Poisson integral of a discrete measure belongs to $h_0(\Delta)$. Since (i) implies (ii), by Lemma 1.7, it remains to prove that (ii) implies (iii) (for $p \leqslant 1/2$, as noted above).

Let $u \in A^p \cap h_0(\Delta)$, $p \leqslant 1/2$. Then, by Lemma 1.8, u is in $A_0^q = h^q(1/q-1)$ for $1/2<q<1$. Hence, by Theorem 2.6, there is a function $\phi \in NBV_q$ such that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta-t)\, d\phi(t).$$

Hence, by partial integration,

$$u(re^{i\theta}) = \text{const.}\, P(r, \theta) + D^1 U(re^{i\theta}),$$

where

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta-t)\, \phi(t)\, dt. \tag{1}$$

Since u and P are in A^p (Lemma 1.7) we see that $D^1U \in A^p$; i.e., $D^nU \in h^p(1/p-n)$ for $n>1/p$. Assuming, as we may, that U is real valued we find that $D^nF \in h^p(1/p-n)$, where F is the holomorphic completion of U (Theorem 1.2(d)). Hence, by Theorem 3.3, $F \in H^p$ and

$$\|\Delta_t^n F^+\|_p = O(t^{1/p}) \qquad (t \to 0^+).$$

And hence, since $\lim_{r\to 1} U(re^{i\theta}) = \phi(\theta)$ a.e. on $[0, 2\pi]$ (by (1)),

$$\|\Delta_t^n \psi\|_p = O(t^{1/p}) \qquad (t \to 0^+), \tag{2}$$

where ψ is 2π-periodic and $\psi(\theta) = \phi(\theta)$ for $0 \leqslant \theta < 2\pi$.

Let S be an arbitrary finite subset of $(0, 2\pi)$. Then we have, by (2),

$$\sum_{x\in S} \int_{x-t}^{x} |\Delta_t^n \psi(\theta)|^p \, d\theta \leqslant Ct \tag{3}$$

for sufficiently small t, where C is a constant independent of S. On the other hand, using the identity

$$\Delta_t^n \psi(\theta) = \phi(\theta) + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \phi(\theta + jt)$$

(valid for $0 < \theta < 2\pi$ and $\theta + nt < 2\pi$) and the left-continuity of ϕ, one easify verifies that

$$\lim_{t\to 0} \frac{1}{t} \int_{x-t}^{x} |\Delta_t^n \psi(\theta)|^p \, d\theta = |\phi(x^+) - \phi(x)|^p$$

for $x \in S$. Combining this with (3) we find that

$$\sum_{x\in S} |\phi(x^+) - \phi(x)|^p \leqslant C,$$

where C is independent of S. Since S is arbitrary we conclude that

$$\sum_{x=0}^{2\pi^-} |\phi(x^+) - \phi(x)|^p < \infty.$$

Now the desired conclusion follows from Lemma 3.4. ∎

4. What Is A^p/A_0^p?

Let $u \in A^p$ (see 1.4). Since A^p is self-conjugate (Theorem 1.2(d)), there exists (by Theorem 3.3) the limit

$$u^+(\theta) = \lim_{r\to 1} u(re^{i\theta}) \qquad (a.e.),$$

and the function u^+ belongs to the Lipschitz class

$$\Lambda^p = \{g : \|\Delta_t^n g\|_p = O(t^{1/p-1})(t \to 0^+)\} \subset L^p.$$

In other, words, the operator B, $Bu=u^+$, maps A^p into Λ^p. And, by our results, the kernel of B is A_0^p. It would be interesting to know whether B is *onto*. A positive answer would imply that the quotient class A^p/A_0^p is canonically isomorphic to Λ^p. (Compare Theorem 3.3.) In connection with this problem we mention Shapiro's result [14] that $h^p(\mathscr{P})/h_0^p = L^p$. Here $h^p(\mathscr{P})$ denotes the closure in $h^p = \{u \in h(\Delta) : M_p(u, r) = O(1)\}$ of the set of harmonic polynomials, and $h_0^p = \{u : \lim M_p(u, r) = 0\}$.

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